**AW3a The probability laws**

In the textbook we informally introduced some of the laws of probability: (a) the probability of an event occurring lies between 0 and 1 (0  P(E)  1), (b) if a sample space comprises 'n' mutually exclusive events, then P(En) = 1, (c) for any two mutually exclusive events A and B within a sample space, then P(A or B) = P(A) + P(B), and (d) for any event within a sample space, P(NOT E) = P(E') = 1 - P(E).

**Example 1**

An experiment consists of tossing three coins. Let events A, B, C, and D represent the events obtained three heads, obtained three tails, obtained only two heads, and obtained only two tails respectively. Figure 1 illustrates the sample space for this experiment and the four mutually exclusive events.



Figure 1

From Figure 1 we have: P(A) = 0.125, P(B) = 0.125, P(C) = 0.375 and P(D) = 0.375. Since the four mutually exclusive events exhaust the sample space then P (A) + P(B) + P(C) + P(D) = 1.0. Since A and B are mutually exclusive, then P(A or B) = P(A) + P(B) = 0.25. Similarly, P(A or B or C) = P(A) + P(B) + P(C) = 0.625. Since P (D) = 0.375, then P(D') = 1 - P(D) = 1 - 0.375 = 0.625.

**The general addition law**

In the above example we demonstrated the addition law for **mutually exclusive events**, P(A or B) = P(A) + P(B). When events are not mutually exclusive, such as, two or more events contain common outcomes within a sample space then this law does not hold.

**Example 2**

To illustrate this case, consider a sample space consisting of the positive integers from 1 through 10. Let event A represent all odd integers and event B represent all integers less than or equal to 5. These two events within the sample space are displayed in Figure 2.



Figure 2

From Figure 2 we note that events A and B overlap with common sample points present. This would be represented by the event {odd integers and integers ≤ 5}. It is important to note that when we ask for the probability of events A and B occurring then this is written as P(A and B). Furthermore, you may see in certain information sources that the mathematical operator (or symbol) ∩ may be used instead of ‘and’. This implies that P(A and B) means the same as P(A ∩ B). The event {A or B} contains the outcomes of either odd integers or integers < 5. A little thought would indicate that the number containing event A or B is given by the equation n{A or B} = n{A) + n{B} - n{A and B}. Consequently, by transforming the events into probabilities the **general addition probability law** is given by equation (1).

P(A or B) = P(A) + P(B) - P(A and B) (1)

If two events are mutually exclusive, P (A and B) = 0.

**Example 3**

A card is chosen from an ordinary pack of cards. Write down the probabilities that the card is: (a) black and an ace, (b) black or an ace, and (c) neither black nor an ace. Let event A and B represent the events obtaining an ace card and B a black card respectively. The sample space is represented by Figure 3.



Figure 3

 (a) P(B and A) = 

(b) P(B or A) = P(B) + P(A) - P(B and A) = 

(c) P(neither B nor A) = 1 - P(B or A) = 1 - 0.5385 = 0.4615

**Conditional probability**

We will now develop the **multiplication law of probability** by considering the concepts of **conditional probability** and **statistical independence**. Consider the differences between choosing an item at random from a lot with and without replacement. Let us say we have 100 items of which 20 are defective and 80 are not defective from which two are selected. Let A = event first item defective and B = event second item defective. If an item is replaced after the first selection the number of items remains constant and P (A) = P (B) =  = 0.2. If the first item is not replaced then what happens to the value of these probabilities?

The probability of event A is still equal to 0.2. In order to determine P(B) we need to know the composition of the lot at the time of the second selection. By not replacing the first item, the total number of items has been reduced to 99, and if the first item is found to be defective then 19 defective items will remain. Thus, the probability of event B occurring, P(B), will now be conditional on whether event A has occurred. This we denote as P(B/A), the conditional probability of the event B given that A has occurred. Thus, for this example P (B/A) =  = 0.1919. In effect, we are computing P(B) with respect to the reduced sample space of A. The following example will make this clear from which we will develop the **multiplication law**.

**Example 4**

Of a group of 30 students, 15 are blue eyed {B}, 5 are left handed {L}, and 2 are both blue eyed and left handed {B and L}. The sample space is represented in Figure 4.



Figure 4

Picking one student at random then the probabilities would be as follows: P(L) = 5/30 ; P(B) = 15/30 and P(L and B) = 2/30. If we know that a student is blue eyed then our sample space will be reduced to 15 students of which 2 are left handed. Thus, P(L/B) = number in {L and B}/number in {B}. By dividing top and bottom by the total sample space gives:



In general, if we have two events A and B then the **probability of event A given that event B has occurred** is given by equation (2).

 (2)

This general result can be converted to give the **multiplication law for joint events** is given by equation (3).

P(A and B) = P(A/B) \* P(B) (3)

**Example 5**

Consider two events A and B which contain all sample points with P(A and B) = 1/4 and P(A/B) = 1/3. Calculate (a) P(B), (b) P(A), and (c) P(B/A).

(a) P(B)? From equation (W4.3.4) we have P(B) = P(A and B)/P(A/B) = (1/4)/(1/3) = 3/4. Therefore, the probability of event B occurring is 0.75 or 75%.

(b) P(A)? Because A and B exhaust the sample space, P(A or B) = 1.0. From the addition law, P(A or B) = P(A) + P(B) - P(A and B). Thus, P(A) = P(A and B) + P(A or B) - P(B) = 1.0 + 0.25 - 0.75 = 0.5. Thus, the probability of event A occurring is 0.5 or 50%.

(c) P(B/A)? P(A and B) is the same as P(B and A). Thus, from the multiplication law, P(A and B) = P(B/A) \* P(A). Re-arranging this equation gives P(B/A) = P(B and A)/P(A) = 0.25/0.5 = 0.5. Thus, the probability that event B occurs given that even A has already occurred is 0.5 or 50%.

**Example 6**

An office is due to be modernised with new office equipment. To aid the office manager a survey has been undertaken to identify the following information: (a) number of laptops, (b) number of desktops, and (c) whether the computers are old or new. The data collected is provided in Table 1.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Laptops, L | Desktops, D | Totals |
| New, N | 40 | 30 | 70 |
| Old, O | 20 | 10 | 30 |
| Totals | 60 | 40 | 100 |

Table 1

If a person picks one computer at random, calculate the following probabilities: (a) computer new, (b) computer a laptop, and (c) the computer new given that it is a laptop. Parts (a) and (b) deal with distinct mutually exclusive events within the full sample space. Hence P(N) = 70/100 = 0.70 and P(L) = 60/100 = 0.60. In part (c) we are dealing with the conditional probability P(N/L). By considering the reduced sample space L (60 laptops) then P(N/L) = 40/60 = 0.66’ or by considering the definition of conditional probability P(N/L) = P(N and L)/P(L) = (40/100)/(60/100) = 0.66’. Both methods will give us the same answer, 66 2/3%, for the probability that it is new given it is a laptop.

**Example 7**

A box contains 6 red and 10 black balls. What is the probability that if three balls are chosen one at a time without replacement that they are all black. Let B1 = Event first draw black, B2 = Event second draw black, and B3 = Event third draw black. In this example we are determining the probability that all three balls chosen are black (P(B1 and B2 and B3)).

On the first draw P(B1) = 10/16. On the second draw the sample space has been reduced to 15 balls and given the condition that the first ball is black then P(B2 / B1) = 9/15. On the third draw the sample space has been reduced to 14 balls and given the condition that the first and second balls are black then P(B3/(B2 and B1)) = 8/14. Thus, P(B1 n B2 n B3) = P(B1) \* P(B2/B1) \* P(B3 / (B2 n B1)) = (10/16) \* (9/15) \* (8/14) = 0.2143. Therefore, the probability that all three balls are black when no replacement occurs is 0.2143 or 21.4%.

**Statistical independence**

We have already considered mutually exclusive events, such that events cannot occur at the same time. We have also noted that in some situations knowing that one event has occurred yields information that will affect the probability of another event. There will be many situations where the converse is true. For example, rolling a die twice, knowing that a six resulted on the first roll cannot influence the outcome of the second roll. Similarly take the example of picking a ball from a bag, if it were replaced before another was picked nothing changes; the sample space remains the same. Drawing the first ball and replacing it cannot affect the outcome of the next selection. In these examples, we have the notion of **independent events**. If two (or more) events are independent, then the **multiplication law for independent events** is given by equation (4).

P(A and B) = P(A) \* P(B) (4)

From which we can deduce that for independent events P(A/B) = P(A) and similarly P(B/A) = P(B). The terms independent and mutually exclusive are different concepts. If A and B are events with non-zero probabilities, then we can show that P(A and B):

* If events A and B are mutually exclusive, then P(A and B) = 0. Mutually exclusive events cannot occur at the same time. For example, the two events ‘my favourite football team lost a match’ and ‘my favourite football team won the same match’ are mutually exclusive events.
* If two events A and B are independent, then P(A and B)  0. The outcome of event A, has no effect on the outcome of event B. For example, the two events ‘it rained in Paris’ and ‘my car broke down in London’ are independent events. When calculating the probabilities for independent events you multiply the probabilities. You are effectively saying, what is the chance of both events happening bearing in mind that the two were unrelated?

So, if events A and B are mutually exclusive, they cannot be independent. If events A and B are independent, they cannot be mutually exclusive.

**Example 8**

Suppose a fair die is tossed twice. Let event A represent the event first die shows an even number and event B represent the event second die shows a 5 or 6. Events A and B are intuitively unrelated and therefore are independent events. Thus the probability of A occurring is P(A) = 3/6 = 1/2 and the probability of event B occurring is P(B) = 2/6 = 1/3. Thus, P(A and B) = P(A) \* P(B) = (1/2) \*(1/3) = 1/6. Thus, probability of events A and B occurring together is 1/6.

**Example 9**

Three marksmen take place in a shooting contest. Their chances of hitting the 'bull' are 1/2, 1/3, 1/4 respectively. If they fire simultaneously what are the chances that only one bullet will hit the 'bull'? Let event A, B, C represent the event that the first man hits the 'bull', second man hits the 'bull', and third man hits the 'bull' respectively, with the following probabilities: P(A) = 1/2 ; P(B) = 1/3 ; P(C) = 1/4. The probability problem can be written as follows:

P(only one bull hit) = P(A and B' and C' OR A' and B and C' OR A' and B' and C)

P(only one bull hit) = P(A and B' and C') + P(A' and B and C') + P(A' and B' and C)

P(only one bull hit) = 1/2 \* 2/3 \* 3/4 + 1/2 \* 1/3 \* 3/4 + 1/2 \* 2/3 \* 1/4 = 1/4 + 1/8 + 1/12

P(only one bull hit) = 11/24.

Thus, the probability that one bull is hit between the three marksmen is 11/24 or 45.83%.

In the solution, we have used the notation A’, B’, and C’. This notation represents the event that the event does not occur, for example, A’ would represent the event that event A does not occur.

**Example 10**

Two football teams A and B are disputing the historical data of who is likely to win. To settle the dispute the following probability data presented in Table 2 has been collected that measures the probability of each team scoring 0, 1, 2, or 3 goals. Calculate the probability: (a) team A wins, (b) teams draw, and (c) team B wins.

|  |  |
| --- | --- |
|  | Number of goals scored |
|  | 0 | 1 | 2 | 3 |
| Team A | 0.3 | 0.3 | 0.3 | 0.1 |
| Team B | 0.2 | 0.4 | 0.3 | 0.1 |

Table 2

To solve this problem, we need to find the total sample space. There are 16 possible results (events) given the above scores, each of which are mutually exclusive. We will look at these in a joint probability table assuming independence i.e. this means that team A scoring does not influence team B scoring (Table 3).

|  |  |  |
| --- | --- | --- |
|  |  | Team A scores |
|  |  | 0 | 1 | 2 | 3 |
| Team B scores | 0 | 0.06 | 0.06 | 0.06 | 0.02 |
| 1 | 0.12 | 0.12 | 0.12 | 0.04 |
| 2 | 0.09 | 0.09 | 0.09 | 0.03 |
| 3 | 0.03 | 0.03 | 0.03 | 0.01 |

Table 3

Since the events are mutually exclusive then the probabilities are as follows:

P(A wins) = 0.06 + 0.06 + 0.02 + 0.12 + 0.04 + 0.03 = 0.33

P(Draw) = 0.06 + 0.12 + 0.09 + 0.01 = 0.28

P(B wins) = 1 - {P(A wins) + P(Draw)} = 1 - {0.33 + 0.28} = 0.39

From these results, we can see that team B has the greater chance of winning a game.

#### Check your understanding

X1

For each question indicate whether the events are mutually exclusive: (a) thermometers are inspected and rejected if any of the following are found: poor calibration; inability to withstand extreme temperatures without breaking; not within specified size tolerances, and (b) a manager will reject a job applicant for any of the following reasons: lack of relevant experience; slovenly appearance; too old.

X2

Consider two events A and B of an experiment which is not empty. Display this information in a Venn diagram and shade the area representing the event {A or B'}.

X3

Consider two events A and B where the associated probabilities are as follows: P(A or B) = 3/4, P(B) = 3/8 and n(A) = 4. Calculate P(A and B) if the total sample size is 8.

X4

A survey shows that 80% of all households have a colour television and 30% have a microwave oven. If 20% have both a colour television and a microwave, what percentage has neither?

X5

In a group of 50 students, 30 study French or German. If 20 study French and 15 studies German find the probability that a student studies French and German.

X6

A bowl contains three red chips and five blue chips. Two chips are drawn successively, at random and without replacement. Calculate the probability that the first chip drawn is red and the second blue?

X7

Two events D and E are found to have the following probability relationships: P(D) = 1/3, P(E) = 1/4, and P(D or E) = 1/2. Calculate the following probabilities: (a) P(D and E), (b) P(D/E), and (c) P(E/D)?

X8

Two events A and B are found to have the following probability relationships: P(A) = 1/3, P(B) = 1/2, and P(A or B) = 3/4. Calculate the following probabilities: (a) P(A/B), (b) P(B/A), (c) P(B'/A'), and (d) P(A'/B')?

X9

A bag contains 4 red counters and 6 black counters. A counter is picked at random from the bag and not replaced. A second counter is then picked. Calculate the following probabilities: (a) the second counter is red, given that the first is red, (b) both the counters are red, and (c) the counters are of different colours?

X10

The Gompertz Oil Company drills for oil in old oil fields that the large companies have stated are uneconomic. The decision to drill will depend upon a number of factors, including the geology of the proposed sites. Drilling experience shows that there is a 0.40 probability of a type A structure present at the site given a productive well. It is also known that 50% of all wells are drilled in locations with a type A structure and 30% of all wells drilled are productive. Use the information provided to answer the following questions: (a) what is the probability of a well drilled in a type A structure and being productive, (b) what is the probability of having a productive well at the location if the drilling process begins in a location with a type A structure, and (c) is finding a productive well independent of the type A structure?

X11

A dart is thrown at a board and is equally likely to land in any one of eight squares numbered 1 to 8 inclusive. Let A = Event dart lands in square 5 or 8, B = Event dart lands in square 2, 3 or 4, and C = Event dart lands in square 1, 2, 5 or 6. From this information calculate the following probabilities: (a) P(A n B), (b) P(A and C), (c) P(B and C), (d) P(A/B), (e) P(B/C), and (f) P(C/A). Which two events are mutually exclusive? Which two events are statistically independent?